Exponentials and logarithms: a creation story

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One of the ways of constructing $\mathbb{R}$ is as a (set of) so-called *Cauchy sequences* of rational numbers. And from here, one can define raising a (positive) real number a to a real exponent x as $a^x \stackrel{\text{def}}{=} \lim_{n\to\infty} a^{x_n}$, where $\{x_n\}$ is a (rational) Cauchy sequence representing the real x.[1] And one way of introducing the exponential function with base e (Napier's number), as well as its inverse (the logarithm with the same base), is to attempt to compute the derivative of the function a^x (a being fixed and x being the independent variable). Attempting to proceed via the classical definition of derivative, we have:

$$(a^x)' = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h\to 0} \frac{(a^h - 1)}{h} \tag{1}$$

This appears to be a dead end, because the expression we computed for the derivative of a^x, is dependent on a^x... However, it turns out that a bit of speculation will take us a long way indeed! Let us begin by assuming that the limit exists—i.e., that $\lim_{h\to 0}(a^h - 1)/h = \beta$. Letting $f(x) = a^x$ (with $f\colon \mathbb{R} \to \mathbb{R}^+$), this means that $f'(x) = \beta \cdot f(x)$.[2]

Now, by the rule for the derivative of the inverse function, we have:

$$\left[f^{-1}\right]'(x) = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{\beta \cdot f\left[f^{-1}(x)\right]} = \frac{1}{\beta x}$$

This of course, assumes that f is invertible, i.e., bijective—and thus, that one can define $f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$. Furthermore, because of the Fundamental Theorem of Calculus, we have:

$$\int_1^x \frac{1}{\beta t} dt = f^{-1}(x) - f^{-1}(1) = f^{-1}(x)$$

where the last equality is because as $f(0) = 1$, $f^{-1}(1) = 0$. The value of β, however, remains unknown—but it stands to reason that it ought to depend on a. Proceeding on that basis, we set $\beta = 1$ and attempt

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to determine the corresponding value of a. Relying on hindsight, let us define

$$\log x \stackrel{\text{def}}{=} \int_1^x \frac{1}{t} dt \qquad \text{and} \qquad \exp x \stackrel{\text{def}}{=} \log^{-1} x \tag{2}$$

where $\log\colon \mathbb{R}^+ \to \mathbb{R}$ and $\exp\colon \mathbb{R} \to \mathbb{R}^+$. Note this means $\log' x = 1/x$—thus establishing that log is continuous. An important clue that we are on the right track is that, if the log function is invertible, then the function exp—which is that inverse, by definition—is, as we expect, its own derivative (which shows that exp is also continuous):

$$\exp' x = (\log^{-1} x)' = \frac{1}{\log'(\log^{-1} x)} = \frac{1}{\log'(\exp x)} = \frac{1}{1/(\exp x)} = \exp x$$

But to show that definitions in (2) are indeed proper, we must show that log is a bijection, and is thus invertible.[3] As $1/y$ is continuous for $y \in \mathbb{R}^+$, the integral $\int_1^x 1/t \, dt$ is well-defined for any $x \in \mathbb{R}^+$—and so, we can define the domain of log to be $\mathbb{R}^+$.[4] As for its range, we will show it is all of $\mathbb{R}$. We begin by observing that as, by definition, $\log' x = 1/x$, and $x > 0$, then $1/x > 0$, i.e., $\log'$ is always positive, which means log is strictly increasing. We now require (the corollary to) the following lemma:

Lemma 3. *Given positive reals* a, b, *we have* $\log a + \log b = \log(ab)$.

Proof. Let $c > 0$ be a real number. We have $\log'(cx) = 1/(cx) \cdot c = 1/x$—meaning $\log'(cx) = \log' x$. Thus there exists a constant k such that, for all x where the derivative of log is defined, we have $\log(cx) = \log x + k$. In particular, for $x = 1$ we obtain: $\log(c \cdot 1) = \log 1 + k \Leftrightarrow \log c = k$. From which $\log(cx) = \log x + \log c$. The fact that c is arbitrary completes the proof. ∎

Corollary 4. *For any positive real* a, *we have* $\log a^{-1} = -\log a$.

Proof. Setting $b = 1/a$ in lemma 3 we have:

$$0 = \log\left(a \times \frac{1}{a}\right) = \log a + \log\frac{1}{a} \Leftrightarrow \log a^{-1} = -\log a$$

∎

Corollary 5. *For any positive real* x, *and integer* n, *we have* $\log x^n = n \log x$.

Proof. For $n = 0$ it is obvious. For $n > 0$, in lemma 3 set $a = b = x$, and use induction on n. For $n < 0$, write $\log x^n$ as $\log(x^{-1})^{-n}$. As $-n > 0$, by the previous induction it follows that this is equal to $-n \log x^{-1}$—and from corollary 4, this equals $n \log x$. ∎

We can now show that the range of log is indeed $\mathbb{R}$, by showing that for any $y \in \mathbb{R}$, the equation $\log x = y$ has always one solution (it cannot have more than one solution, because that would mean it was not injective—and thus not strictly increasing). To show this is so, by the definition of log, we clearly have $\log 2 > 0$—and thus, $y/(\log 2)$ is finite. Hence, we can find integers m, n such that $m < y/(\log 2) < n \Leftrightarrow m \log 2 < y < n \log 2 \Leftrightarrow \log 2^m < y < \log 2^n$. By the Intermediate Value Theorem, there exists $x \in]2^m, 2^n[$ such that $\log x = y$.

This shows that log is indeed a bijection—and thus, that definitions (2) are proper. But we want to go further: log was defined as the putative inverse of a function of the form a^x, for some positive real a, and exp being the inverse of log, we want to express it in this form as well (i.e., we want to find a real a such that $\exp x = a^x$). This requires the following lemma.

Lemma 6. *Let* f *be a continuous function, for which it holds that* $f(x+y) = f(x)f(y)$*,* $f(0) = 1$ *and* $f(1) = a > 0$*. Then* $f(x) = a^x$*.*

Proof. Let us first prove that it holds for positive integers (let n be one such integer):

$$f(n) = f(\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}) = \underbrace{f(1) \cdots f(1)}_{n \text{ times}} = a^n$$

Now let us show that also holds for negative integers:

$$\begin{aligned} 1 = f(0) &= f(n + (-n)) = f(n)f(-n) \\ \Leftrightarrow f(-n) &= \frac{1}{f(n)} = \frac{1}{a^n} = (a^n)^{-1} = a^{-n} \end{aligned}$$

And now for rational numbers, let m, n be integers, with $n > 0$. We have:

$$\begin{aligned} a^m = f(m) &= f\Bigg(\underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{n \text{ times}}\Bigg) = \underbrace{f\left(\frac{m}{n}\right) \cdots f(\frac{m}{n})}_{n \text{ times}} \\ \Leftrightarrow f\left(\frac{m}{n}\right) &= \sqrt[n]{a^m} = a^{m/n} \end{aligned}$$

And finally, for real numbers, let x be a real, and $\{x_n\}$ be a Cauchy sequence of rational terms, such that $x_n \to x$. Because f is continuous, we have $f(x) = f(\lim_{n \to +\infty} x_n) = \lim_{n \to +\infty} f(x_n) = \lim_{n \to +\infty} a^{x_n}$ which is, by definition, a^x. ∎

We have already established that $\exp 0 = 1$—and so, for exp to verify the conditions of lemma 6, it remains only to show that $\exp 1 > 0$ and $\exp(x + y) = \exp x \cdot \exp y$.

To compute $\exp 1$, $\exp$ being indefinitely differentiable and having all the derivatives be continuous as well—recall that $\exp' = \exp$—means that its Taylor series at point $x = 0$ converges for all x in its domain (as $\exp 0 = 1$, all the derivatives at $x = 0$ equal 1):[5]

$$\exp x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Setting $x = 1$, it is now immediate that $\exp 1 = 1 + 1 + 1/2 + 1/6 + \dots$ is a positive real number, that we shall denote by e.

To show that $\exp(x + y) = \exp x \cdot \exp y$, let $x' = \exp x \Leftrightarrow x = \log x'$ and $y' = \exp y \Leftrightarrow y = \log y'$. We have:

$$\begin{aligned} \exp(x + y) &= \exp(\log x' + \log y') \\ &= \exp(\log(x'y')) \qquad \text{(lemma 3)} \\ &= x'y' = \exp x \cdot \exp y \end{aligned}$$

Hence by lemma 6 we conclude that $\exp x = e^x$. And we are (almost) ready to return to our initial goal of differentiating a^x—but first, we need the following generalization of corollary 5:

Lemma 7. *For any positive real a and real b, we have* $\log a^b = b \log a$.

Proof. $\log a^b = \log \left[\left(e^{\log a} \right)^b \right] = \log \left(e^{b \log a} \right) = b \log a.$ ∎

It is now straightforward that:

$$(a^x)' = \left[\left(e^{\log a} \right)^x \right]' = \left(e^{x \log a} \right)' = e^{x \log a} \cdot \log a = a^x \cdot \log a \qquad (8)$$

This also shows that just like $\exp$, a^x is also continuous over all of $\mathbb{R}$. Comparing this with (1), it is immediate that

$$\lim_{h \to 0} \frac{(a^h - 1)}{h} = \log a$$

In particular, if $a = e$, we obtain the well-known limit

$$\lim_{x \to 0} \frac{(e^x - 1)}{x} = \log e = 1$$

Base of a logarithm. Just as $\log$ is the inverse of $\exp x = e^x$, we can have logarithms that are inverses for exponentials with other bases other than e. In particular, the inverse of b^x is denoted $\log_b x$. We have the following "change of base" property:

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad (9)$$

Why this holds is straightforward: $\log_b a \cdot \log_c b = \log_c b^{\log_b a} = \log_c a$. And from (9) it can be easily shown that lemmas 3 and 7 also hold for logarithms with arbitrary bases:

- $\log_c a + \log_c b = \frac{\log a}{\log c} + \frac{\log b}{\log c} = \frac{\log a + \log b}{\log c} = \frac{\log(ab)}{\log c} = \log_c(ab)$
- $\log_c a^b = \frac{\log a^b}{\log c} = \frac{b \log a}{\log c} = b\frac{\log a}{\log c} = b \log_c a$

Finally, for completeness, just as in (8) we computed the derivative of an exponential function with an arbitrary base, here is the derivative for a logarithm with an arbitrary base:

$$(\log_b x)' = \left(\frac{\log x}{\log b}\right)' = \frac{1}{x \log b}$$

Thus, just like log, the $\log_b$ function is also continuous over all its domain.

Notes

1. I have a forthcoming manuscript on the construction of (*inter alia*) the real numbers, using Cauchy sequences—and I will update this note once it is published—to which the reader is (to be) referred for more details. For the reason why a is required to be positive, see §3 and §4 in my essay on exponentiation rules, `https://randomwalk.eu/scholarship/exponentiation-rules-reals/`.
2. Cf. note 1, in particular §3 and §4 of `Exponentiation.in.R.pdf`, for the reasons why the range of f is $\mathbb{R}^+$, i.e., why a^x is always positive.
3. Note that all the assumptions made above—in particular in (1), that the limit $\lim_{h\to 0}(a^h - 1)/h$ existed—are now irrelevant. They were an aid to help us arrive at putative definition (2), but play no role in establishing that it is a proper definition. However, it will, in due time, be shown that they are all correct—and in particular, we will show that the referred limit does, in fact, exist.
4. Note that by this definition $\log x < 0$ for $0 < x < 1$, $\log 1 = 0$, and $\log x > 0$ for $x > 1$.
5. I omit here the statements and proofs of the relevant theorems. But see, e.g., §3 (and the references therein) in my manuscript on the sine and cosine functions, `https://randomwalk.eu/scholarship/sine-cosine/`.